Almost sure functional limit theorem for the product of partial sums

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Abstract. We prove an almost sure functional limit theorem for the product of partial sums of i.i.d. positive random variables with finite second moment.

1. Introduction and Main Result

Limiting distributions of the product of partial sums of positive random variables have been widely studied in recent years. Arnold and Villaseñor [1] proved the limit theorem for the partial sum of a sequence of exponential random variables. Rempała and Wesołowski [4] proved it for any independent and identically distributed (i.i.d.) random variables with finite variance. Later, Qi [5] considered a sequence of random variables with α-stable distribution and established the limit distribution of the product of the partial sums when $1 < \alpha \le 2$.

Recently, Zhang and Huang [6] proved the following invariance principle of the product of partial sums of i.i.d. positive random variables with mean $\mu > 0$ and variance σ^2:

$$\left(\prod_{k=1}^{[nt]} \frac{S_k}{\mu k}\right)^{\frac{\mu}{\sigma\sqrt{n}}} \xrightarrow{\mathcal{D}} \exp\left(\int_0^t \frac{W(s)}{s} ds\right) \text{ as } n \to \infty. \tag{1.1}$$

The goal of this paper is to obtain an almost sure version of the above invariance principle which can also be a functional version of the almost sure limit theorem obtained by Gonchigdanzan and Rempała [3]. Here is the result:

Theorem 1. *Let $(X_k)_{k\ge1}$ be a sequence of i.i.d. positive random variables with mean $\mu > 0$ and variance σ^2 and let $S_n = X_1 + \cdots + X_n$. Define a process $\{\xi_n(t) : 0 \le t \le 1\}$ by*

$$\xi_n(t) := \left(\prod_{k=1}^{[nt]} \frac{S_k}{\mu k}\right)^{\frac{\mu}{\sigma\sqrt{n}}}.$$

Let F_t be the distribution function of the random variable on the right-hand side of (1.1). Then

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\mathbf{I}\left(\xi_k(t) \le x\right) \xrightarrow{a.s.} F_t(x) \text{ as } n \to \infty \tag{1.2}$$

if and only if

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\mathbf{P}\left(\xi_k(t) \le x\right) \longrightarrow F_t(x) \text{ as } n \to \infty. \tag{1.3}$$

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Corollary. *Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. positive random variables with mean $\mu > 0$ and variance σ^2. Then we have*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{I}\left(\left(\prod_{j=1}^{[kt]}\frac{S_j}{\mu j}\right)^{\dfrac{\mu}{\sigma\sqrt{k}}}\leq x\right)\xrightarrow{a.s.} F_t(x)\ \ as\ \ n\to\infty.$$

2. Auxiliary Results and Proofs

Throughout the paper $\log x$ and $\log\log x$ stand for $\ln(e\vee n)$ and $\ln\ln(n\vee e^e)$ respectively, and " $\ll$ " is meant for the big "O" notation.

2.1 Auxiliary Results

Lemma 1. *Let $(Y_n)_{n\geq 1}$ be a sequence of random variables. Set $S_n = Y_1+\cdots+Y_n$. Then we have*

$$E\biggl(\max_{1\leq k\leq n}\Bigl|\sum_{j=1}^{k}\log\left(\frac{n+1}{j}\right)\,Y_j\Bigr|\biggr)\leq 3\log(n+1)\,E\biggl(\max_{1\leq k\leq n}|S_k|\biggr).$$

Proof. Observe that

$$\Bigl|\sum_{j=1}^{k}\log\Bigl(\frac{n+1}{j}\Bigr)Y_j\Bigr|\leq\Bigl|\sum_{j=1}^{k}\log(n+1)\,Y_j\Bigr|+\Bigl|\sum_{j=2}^{k}\log j\,Y_j\Bigr| = T_1+T_2$$

Obviously, $T_1\leq\log(n+1)|S_k|$ which implies

$$\max_{1\leq k\leq n}T_1\leq\log(n+1)\max_{1\leq k\leq n}\bigl|S_k\bigr|.$$

For the second term T_2 we have

$$\begin{aligned}\max_{2\leq k\leq n}T_2 &= \max_{2\leq k\leq n}\Bigl|\sum_{j=2}^{k}\log j\,Y_j\Bigr| = \max_{2\leq k\leq n}\Bigl|\sum_{j=2}^{k}(Y_j+Y_{j+1}+\cdots+Y_k)(\log j-\log(j-1))\Bigr|\\ &\leq\max_{2\leq k\leq n}\sum_{j=2}^{k}|Y_j+Y_{j+1}+\cdots+Y_k|\,(\log j-\log(j-1))\\ &\leq 2\max_{2\leq k\leq n}|Y_1+Y_2+\cdots+Y_k|\sum_{j=2}^{n}(\log j-\log(j-1))\leq 2\log(n+1)\max_{1\leq k\leq n}|S_k|.\quad\square\end{aligned}$$

Lemma 2. *Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. positive random variables with mean μ and variance σ^2. Then setting $S_n = X_1+\cdots+X_n$ we have*

$$\max_{0\leq t\leq 1}\Bigl|\frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\log\frac{S_k}{\mu k}-\frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)\Bigr|\xrightarrow{a.s.}0\ \ as\ \ n\to\infty.$$

Proof. Note that $\log(x+1) = x - r(x)$ where $r(x)/x^2\to\frac{1}{2}$ as $x\to 0$. By the strong law of large numbers we have $S_k/k-\mu\xrightarrow{a.s.}0$ as $k\to\infty$.

Thus by the law of iterated logarithm we get

$$\left|\frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\log\frac{S_k}{\mu k}-\frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)\right|\overset{a.s.}{\ll}\frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(\frac{S_k}{k}-\mu\right)^2$$

$$\overset{a.s.}{\ll}\max_{0\le t\le 1}\frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\frac{1}{k}\log\log k\ll\frac{1}{\sigma\sqrt{n}}\log n\log\log n\to 0.\quad\square$$

2.2 Proof of Theorem 1.

We use Berkes and Dehling's [2] technique to prove our theorem. Observe that

$$\sum_{k=1}^{n}\left(\frac{S_k}{k}-\mu\right)=\sum_{k=1}^{n}b_{k,n}(X_k-\mu)$$

where $b_{k,n}=\sum_{j=k}^{n}1/j$. Hence by Lemma 2 it suffices to show that for any Borel-subset A of $D[0,1]$

$$\frac{1}{\log n}\sum_{k=2}^{n}\frac{1}{k}\mathbf{I}\left(\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\in A\right)-\mathbf{P}\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\in A\right)\right)\xrightarrow{a.s.}0\ \text{ as }\ n\to\infty.\tag{2.1}$$

where $\hat{s}_n=\sum_{i=1}^{[nt]}b_{i,[nt]}(X_i-\mu)$. From Berkes and Dehling [2] (p.1647), to prove (2.1) it suffices to show that for any bounded Lipschitz function f on $D[0,1]$ we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\zeta_k\xrightarrow{a.s.}0\ \text{ as }\ n\to\infty\tag{2.2}$$

where $\zeta_k=f(\hat{s}_k/\sigma\sqrt{k})-\mathbf{E}f(\hat{s}_k/\sigma\sqrt{k})$. In fact, the following implies (2.2) (see p.1648, Berkes and Dehling (1993) for the proof):

$$\mathbf{E}\left(\sum_{k=1}^{n}\frac{1}{k}\zeta_k\right)^2\ll\log^2 n(\log\log n)^{-\varepsilon}\ \text{ for some }\ \varepsilon>0.\tag{2.3}$$

Therefore, showing (2.3) would be sufficient for the proof of Theorem 1. Observing

$$\hat{s}_l-\hat{s}_k=b_{[kt]+1,[lt]}(S_{[kt]}-[kt]\mu)+\big(b_{[kt]+1,[lt]}(X_{[kt]+1}-\mu)+\cdots+b_{[lt],[lt]}(X_{[lt]}-\mu)\big)$$

for $l\ge k$ we find that $\hat{s}_l-\hat{s}_k-b_{[kt]+1,[lt]}(S_{[kt]}-\mu[kt])$ is independent of $\hat{s}_{[kt]}$ which implies that

$$Cov\left(f\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\right),f\left(\frac{\hat{s}_l-\hat{s}_k-b_{[kt]+1,[lt]}(S_{[kt]}-\mu[kt])}{\sigma\sqrt{l}}\right)\right)=0\ \text{ for }\ l\ge k.$$

By the Lipschitz property of f

$$\begin{aligned}\left|\mathbf{E}(\zeta_k\zeta_l)\right|&\ll\left|Cov\left(f\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\right),f\left(\frac{\hat{s}_l}{\sigma\sqrt{l}}\right)-f\left(\frac{|\hat{s}_l-\hat{s}_k-b_{[kt]+1,[lt]}(S_{[kt]}-\mu[kt])|}{\sigma\sqrt{l}}\right)\right)\right|\\&\ll\mathbf{E}\left(\max_{0\le t\le 1}\frac{|\hat{s}_k+b_{[kt]+1,[lt]}(S_{[kt]}-\mu[kt])|}{\sigma\sqrt{l}}\right)\\&\ll\mathbf{E}\left(\max_{0\le t\le 1}\frac{|\hat{s}_k|}{\sigma\sqrt{l}}\right)+\mathbf{E}\left(\max_{0\le t\le 1}\frac{|b_{[kt]+1,[lt]}(S_{[kt]}-\mu[kt])|}{\sigma\sqrt{l}}\right)\\&\ll\left(\frac{k}{l}\right)^{1/2}\mathbf{E}\left(\max_{0\le t\le 1}\frac{|\hat{s}_k|}{\sigma\sqrt{k}}\right)+\left(\frac{k}{l}\right)^{1/2}\mathbf{E}\left(\max_{0\le t\le 1}b_{[kt]+1,[lt]}\frac{|S_{[kt]}-\mu[kt]|}{\sigma\sqrt{k}}\right).\end{aligned}$$

Since $\max_{0\le t\le 1} b_{[kt]+1,[lt]} = \log(l/k) \ll (l/k)^{\gamma}$ (we choose $0<\gamma<1/2$)

$$\begin{aligned}\left|\mathbf{E}(\zeta_k\zeta_l)\right| &\ll \left(\frac{k}{l}\right)^{1/2}\mathbf{E}\left(\max_{0\le t\le 1}\frac{1}{\sigma\sqrt{k}}\left|\sum_{i=1}^{[kt]} b_{i,k}(X_i-\mu)\right|\right)+\left(\frac{k}{l}\right)^{1/2-\gamma}\mathbf{E}\left(\max_{0\le t\le 1}\frac{|S_{[kt]}-\mu[kt]|}{\sigma\sqrt{k}}\right)\\ &= \left(\frac{k}{l}\right)^{1/2}\mathbf{E}\left(\max_{0\le j\le k}\frac{1}{\sigma\sqrt{k}}\left|\sum_{i=1}^{j} b_{i,k}(X_i-\mu)\right|\right)+\left(\frac{k}{l}\right)^{1/2-\gamma}\mathbf{E}\left(\max_{0\le j\le k}\frac{|S_j-\mu j|}{\sigma\sqrt{k}}\right)\\ &= M_1+M_2.\end{aligned}$$

Now applying Lemma 1 to M_1 we obtain

$$\begin{aligned}\left|\mathbf{E}(\zeta_k\zeta_l)\right| &\ll \left(\frac{k}{l}\right)^{1/2}\log k\,\mathbf{E}\left(\max_{1\le j\le k}\frac{1}{\sigma\sqrt{k}}\left|\sum_{i=1}^{j}(X_i-\mu)\right|\right)+\left(\frac{k}{l}\right)^{1/2-\gamma}\mathbf{E}\left(\max_{1\le j\le k}\frac{|S_j-\mu j|}{\sigma\sqrt{k}}\right)\\ &\ll \left|\mathbf{E}(\zeta_k\zeta_l)\right| \ll \log k\,\left(\frac{k}{l}\right)^{1/2-\gamma}\mathbf{E}\left(\max_{1\le j\le k}\frac{|S_j-\mu j|}{\sigma\sqrt{k}}\right)\ll \log k\,\left(\frac{k}{l}\right)^{\gamma'}\end{aligned}$$

where $0<\gamma'<1/2-\gamma$.

On the other hand $\mathbf{E}(\zeta_k\zeta_l)\ll 1$ because ζ_k is bounded . Hence we have the following estimate for $\mathbf{E}(\zeta_k\zeta_l)$:

$$\mathbf{E}(\zeta_k\zeta_l)\ll\begin{cases}1, & \text{if } l/k\le \exp\big((\log n)^{1-\varepsilon}\big)\\ (k/l)^{\gamma'}\log k, & \text{if } l/k\ge \exp\big((\log n)^{1-\varepsilon}\big)\end{cases}$$

where ε is any positive number. Hence,

$$\sum_{\substack{1\le k\le l\le n\\ l/k\le \exp{(\log n)^{1-\varepsilon}}}}\frac{\mathbf{E}(\zeta_k\zeta_l)}{kl}\le \sum_{1\le k\le n}\frac{1}{k}\sum_{k\le l\le ke^{(\log n)^{1-\varepsilon}}}\frac{1}{l}\ll\sum_{k=1}^{n}\frac{1}{k}\log^{1-\varepsilon} n\ll \log^{2-\varepsilon} n \tag{2.4}$$

and

$$\sum_{\substack{1\le l\le k\le n\\ l/k\ge \exp{(\log n)^{1-\varepsilon}}}}\frac{\mathbf{E}(\zeta_k\zeta_l)}{kl}\le e^{-\gamma'(\log n)^{1-\varepsilon}}\log n\sum_{1\le k\le l\le n}\frac{1}{kl}\ll e^{-\gamma'(\log n)^{1-\varepsilon}}\log^3 n\ll \log^{2-\varepsilon} n. \tag{2.5}$$

Thus (2.4) and (2.5) immediately imply (2.3) which completes the proof of Theorem 1. □

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